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Theoretical and numerical analysis of fracture phenomena under dynamic loading

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1 Introduction

Let us consider the linear elasticity in the state of plate stress regarding $(x_1, x_2) \in \Omega_\Sigma = \Omega \setminus \Sigma$. Here Ω is the bounded domain in $\mathbb{R}^2$ with a smooth boundary containing the crack whose undeformed shape is a piecewise smooth curve $\Sigma = \sum_{j=1}^{J} \Sigma_j$ in $\mathbb{R}^2$ with the two edges γ_0, γ, which is given by

$$\begin{aligned} \Sigma &= \{(x_1, x_2) |\, x_1 = \phi_1(s),\, x_2 = \phi_2(s),\, 0 \le s \le a\} \\ \Sigma_j &= \{(x_1, x_2) |\, x_1 = \phi_1(s),\, x_2 = \phi_2(s),\, a_j \le s \le a_{j+1}\} \quad a = a_{J+1} \end{aligned}$$

with the length parameter s. The functions $\phi_l(s)$, $l = 1, 2$ are C^2 functions in the interval (a_j, a_{j+1}), $j = 1, \cdots, J$ and the edges are given by $\gamma_0 = (\phi_1(0), \phi_2(0))$, $\gamma = (\phi_1(a), \phi_2(a))$. We assume $\Sigma \subset \Omega$.

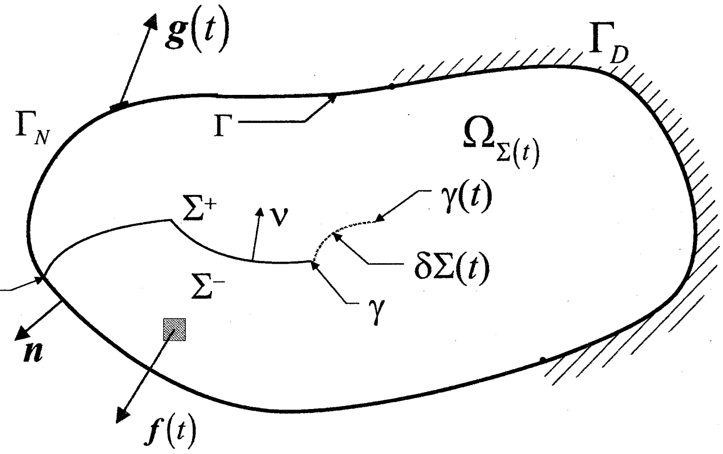

Figure 1: The elastic plate with the crack $\Sigma(t)$.

The crack extension process is considered to occur in a quasi-static manner such that inertial effects may be neglected. Therefore, when we refer to time t, we use it as a

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parameter that delineates the history of the sequence of events such as in loading or crack propagation.

Assume that the part Γ_N of the boundary $\partial\Omega$ is fixed and loads $t \mapsto \boldsymbol{f}(t) \in C^2([0,T), L^2(\mathbb{R}^2)^2)$ and $t \mapsto \boldsymbol{g}(t) \in C^2([0,T), L^2(\Gamma_N)^2)$ are given. In this situation (see Figure 1), we consider the virtual crack extension $\{\Sigma(t)\}_{0 \le t \le T}$

$$\begin{aligned} \Sigma(T) &= \Sigma \cup \delta\Sigma(T), \quad \delta\Sigma(T) \subset \Omega, \quad \Sigma(t) = \Sigma \cap \delta\Sigma(t),\ 0 < t < T, \\ \delta\Sigma(t) &= \{(x_1, x_2) |\ x_1 = \phi_1(s),\ x_2 = \phi_2(s),\ a \le s \le a + t\}, \end{aligned}$$

where $\phi_i(t)$, $i = 1, 2$ are C^2 class in $(0, T)$. For simplify, we assume the parameter t also express the length of crack extension. Throughout this paper the unit vector $\boldsymbol{\nu}(s)$, $s \in (0,T)$ denotes the normal direction from the minus side to the plus side; i.e., $\boldsymbol{\nu}(s) = (-\phi_2'(s), \phi_1'(s))/\sqrt{\phi_1'(s)^2 + \phi_2'(s)^2}$ when x approaches to $(\phi_1(s), \phi_2(s))$ from above $\Sigma(T)$ (denoted by $\Sigma(T)^+$) and $-\boldsymbol{\nu}(s)$ is the interior normal direction from below $\Sigma(T)$ (denoted by $\Sigma(T)^-$).

Let $\boldsymbol{u}(t) = (u_i(t))$, $\boldsymbol{\varepsilon} = (\varepsilon_{ij}(t))$ and $\boldsymbol{\sigma} = (\sigma_{ij}(t))$ denote that displacement vector, the strain tensor and the stress tensor, respectively. The strain-displacement relation is given by

$$\varepsilon_{ij}(t) = \varepsilon_{ij}(\boldsymbol{u}(t)) := (u_{i,j}(t) + u_{j,i}(t))/2, \qquad u_{i,j}(t) = \partial u_i(t)/\partial x_j$$

and the stress and the strain is connected by Hooke's tensor whose components c_{ijkl} $(i, j, k, l = 1, 2)$ are the C^2 functions defined on $\mathbb{R}^2$ satisfying the conditions; $c_{ijkl} = c_{jilk} = c_{klij}$, i.e.

$$\sigma_{ij}(t) = \sigma_{ij}(\boldsymbol{u}(t)) := c_{ijkl}\varepsilon_{kl}(\boldsymbol{u}(t)).$$

For each load $\mathcal{L}(t) = (\boldsymbol{f}(t), \boldsymbol{g}(t))$, $0 \le t \le T$, the displacement $\boldsymbol{u}(t)$ satisfy the following

$$\mathrm{P}_{\mathcal{L}(t),\Sigma(t)} : \quad \begin{cases} -\sigma_{ij,j}(\boldsymbol{u}(t)) = f_i(t) & \text{in } \Omega_{\Sigma(t)} \\ \sigma_{ij}(\boldsymbol{u}(t))^+\nu_j = \sigma_{ij}(\boldsymbol{u}(t))^-\nu_j = 0 & \text{on } \Sigma(t) \\ \sigma_{ij}(\boldsymbol{u}(t))n_j = g_i(t) & \text{on } \Gamma_N \\ \boldsymbol{u}(t) = 0 & \text{on } \Gamma_D \end{cases}$$

where $\sigma_{ij}(\boldsymbol{u}(t))^\pm$ are the value of $\sigma_{ij}(\boldsymbol{u}(t))$ on the plus/minus side of $\Sigma(t)$, $\boldsymbol{n} = (n_1, n_2)$ denote the outward unit normal of $\partial\Omega$.

In next section, for virtual crack extensions, we will show what's the *crack extension force* which motivate and control the deformations associated with crack extension based on the concept written in [3] from mathematical viewpoint. In section 3, we will consider how to select the real crack extension from virtual crack extensions, especially the direction of the crack extension.

2 The crack extension force in brittle fracture

The weak solution $\boldsymbol{u}(t)$ of problem $\mathrm{P}_{\mathcal{L}(t),\Sigma(t)}$ is given as the element of $V(\Omega_{\Sigma(t)})$ minimizing the potential energy functional

$$\mathcal{E}(\boldsymbol{v}; \mathcal{L}(t), \Omega_{\Sigma(t)}) := \int_{\Omega_{\Sigma(t)}} \{w(x, \boldsymbol{v}) - \boldsymbol{f}(t) \cdot \boldsymbol{v}\} dx - \int_{\Gamma_N} \boldsymbol{g}(t) \cdot \boldsymbol{v}\, dl$$

over $\boldsymbol{v} \in V(\Omega_{\Sigma(t)})$. Here $w(x, \boldsymbol{v}) = c_{ijkl}(x)\varepsilon_{ij}(\boldsymbol{v})\varepsilon_{kl}(\boldsymbol{v})/2$,

$$V(\Omega_{\Sigma(t)}) = \{\boldsymbol{v} \in H^1(\Omega_{\Sigma(t)})^2;\ \boldsymbol{v} = 0 \quad \text{on } \Gamma_D\}.$$

Theorem 2.1 *If the line measure of Γ_N is not zero and there is a constant $c_0 > 0$ such that*

$$c_{ijkl}\xi_{ij}\xi_{kl} \geq c_0\xi_{ij}\xi_{ij} \qquad \textit{for all } \xi_{ij} \in \mathbb{R};\ i, j = 1, 2,$$

then the only one solution $\boldsymbol{u}(t)$ exists and satisfy

$$a_{\Omega_\Sigma(t)}(\boldsymbol{u}(t), \boldsymbol{v}) = \int_{\Omega_{\Sigma(t)}} \boldsymbol{f}(t) \cdot \boldsymbol{v} dx + \int_{\Gamma_N} \boldsymbol{g}(t) \cdot \boldsymbol{v} dl \quad \textit{for all } \boldsymbol{v} \in V(\Omega_\Sigma(t)) \tag{1}$$
$$\textit{by } a_{\Omega_\Sigma(t)}(\boldsymbol{u}(t), \boldsymbol{v}) = \int_{\Omega_\Sigma} \sigma_{ij}(\boldsymbol{u}(t))\varepsilon_{ij}(\boldsymbol{v}) dx,$$

and satisfy the conditions in $P_{\mathcal{L}(t),\Sigma(t)}$. Here the boundary conditions in $P_{\mathcal{L}(t),\Sigma(t)}$ has the meaning on $\Sigma(t)$ by $\sigma_{ij}(\boldsymbol{u}(t))^{\pm}\nu_j$ belonging in the dual space of $H_{00}^{1/2}(\Sigma(t))$ with the norm

$$||\varphi||_{1/2,00,\Sigma(t)} = \left(\int_{\Sigma(t)} \rho_{\partial\Sigma(t)}\varphi^2 d\ell\right)^{1/2},$$

where $\rho_{\partial\Sigma(t)}(x)$ is the distance between $x \in \Sigma(t)$ and $\gamma(t)$.

Proof. Refer to [8, 9, 11] for the proof.

Theorem 2.2 (see e.g. [5, 13, 4]) *At each crack tip $\gamma(t)$, let us consider the local polar coordinates (r_t, θ_t) that are oriented in such a way that the tangent half-line on the side of crack surfaces correspond to the angles $\{-\pi, +\pi\}$, that is, the plus side is on $+\pi$ and the minus side is on $-\pi$. If the elasticity is homogeneous isotropic, then we get the expansion on the neighborhood $U(\gamma(t))$ of the crack tip $\gamma(t)$, $t > 0$,*

$$u_i(x, t) = \sum_{m=1}^{2} \frac{K_m(\gamma(t))}{2\mu}\sqrt{\frac{r_t}{2\pi}}S_{i,m}^C(\theta_t) + u_{i,R}(x, t)$$

for $x = (r_t, \theta_t) \in U(\gamma(t))$, where $u_i(x, t)$ are the components of the displacement $\boldsymbol{u}(t)$ and $u_{i,R}(t) \in H^2(U(\gamma(t)))$.

In the case of the constant load $\mathcal{L} = (\boldsymbol{f}, \boldsymbol{g})$, the most important parameter in fracture mechanics is the *energy release rate*

$$\mathcal{G}(\mathcal{L}; \Omega_{\Sigma(\cdot)}) := \lim_{t\downarrow 0} t^{-1}\left[\mathcal{E}(\boldsymbol{u}(0); \mathcal{L}, \Omega_{\Sigma(0)}) - \mathcal{E}(\boldsymbol{u}(t); \mathcal{L}, \Omega_{\Sigma(t)})\right],$$

which is the derivative $-d\mathcal{E}(\boldsymbol{u}(t); \mathcal{L}(t), \Omega_{\Sigma(t)})/dt|_{t=+0}$ with respect to the crack extension. Under the constant loading, we have the following.

2.1 Generalized J-integral

Let ω be a bounded domain in $\mathbb{R}^2$. We call the domain ω "regular relative to Ω_Σ" if the identity

$$\begin{aligned}\int_{\omega\cap\Omega_\Sigma} vw_{,i}dx &= -\int_{\omega\cap\Omega} v_{,i}wdx + \int_{\partial(\omega\cap\Omega)} vw\, n_i ds \\ &\quad + \int_{\omega\cap\Sigma} \left(v^+w^+\,\nu_i - v^-w^-\,\nu_i\right) ds\end{aligned} \tag{2}$$

holds for all $v,\ w \in H^1(\Omega_\Sigma)$ and each $i = 1, 2$. If $\omega \cap \Omega_\Sigma$ has the local Lipschitz property, then (2) holds (see e.g. [6, p.121]). Therefore $\omega\cap\Omega_\Sigma$ is regular relative to Ω, if $\omega\cap\Omega_\Sigma$ can be decomposed into two disjoint domains $(\omega\cap\Omega_\Sigma)^\pm$ such that $\omega\cap\Sigma \subset \overline{(\omega\cap\Omega)^+}\cap\overline{(\omega\cap\Omega)^-}$.

For each solution $\boldsymbol{u}$ of $\mathrm{P}_{\mathcal{L},\Sigma}$, we define the GJ-integral by

$$J_\omega(\boldsymbol{u}, \mathcal{X}) = P_\omega(\boldsymbol{u}, \mathcal{X}) + R_\omega(\boldsymbol{u}, \mathcal{X})$$

as a functional depending on the domain ω and a vector field $\mathcal{X} \in W^{1,\infty}(\mathbb{R}^2)^2$, where

$$\begin{aligned}P_\omega(\boldsymbol{u}, \mathcal{X}) &= \int_{\partial(\omega\cap\Omega)} \{w(\boldsymbol{u})(\mathcal{X}\cdot\boldsymbol{n}) - \sigma_{ij}(\boldsymbol{u})n_j(\mathcal{X}\cdot\nabla u_i)\}\, d\ell, \\ R_\omega(\boldsymbol{u}, \mathcal{X}) &= \int_{\omega\cap\Omega_\Sigma} \{\sigma_{ij}(\boldsymbol{u})\partial_j\mathcal{X}_k\partial_k u_i - \boldsymbol{f}(\mathcal{X}\cdot\nabla u) - \boldsymbol{X}\cdot\nabla_x w(x,\boldsymbol{u}) - w(x,\boldsymbol{u})\mathrm{div}\mathcal{X}\}\, dx,\end{aligned} \tag{3}$$

are well-defined. Here $\boldsymbol{n} = (n_1, n_2)$ is the outward unit normal on $\partial(\omega\cap\Omega)$ and $d\ell$ the line element of $\partial(\omega\cap\Omega)$. The integral $R_\omega(\boldsymbol{u}, \mathcal{X})$ is well-defined for all solution $\boldsymbol{u}$ of $\mathrm{P}_{\mathcal{L},\Sigma}$, but $P_\omega(\boldsymbol{u}, \mathcal{X})$ needs the regularity of $\boldsymbol{u}$. We notice that $P_\omega(\boldsymbol{u}, \mathcal{X})$ contains no integral over $\omega\cap\Sigma$.

Theorem 2.3 (refer to [7, 8, 10]) *If the virtual crack extension $\{\Sigma(t)\}_{0\le t\le T}$ is smooth at γ, ,i.e.*

$$\phi_i(s),\ i = 1, 2 \quad \textit{are } C^2 \textit{ class on } a_J < s < a + T \quad (a = a_{J+1}),$$

then we have

$$\mathcal{G}(\mathcal{L}, \Omega_{\Sigma(\cdot)}) = J_\omega(\boldsymbol{u}; \boldsymbol{X}), \tag{4}$$

where ω stands for an arbitrary small domain containing the crack tip γ and $\boldsymbol{X}$ the vector field obtained from parallel extension of $\boldsymbol{X}_\gamma = (d\phi_1(s)/ds, d\phi_2(s)/ds)_{s=a}$ over $\overline{\omega}$. Moreover, we have

$$\begin{aligned}\mathcal{G}(\mathcal{L}, \Omega_{\Sigma(\cdot)}) &= \boldsymbol{X}_\gamma\cdot\boldsymbol{J}_\gamma(\boldsymbol{u}) \\ \boldsymbol{J}_\gamma(\boldsymbol{u}) &= \left(\lim_{|\omega|\to 0} P_\omega(\boldsymbol{u}, \boldsymbol{e}_1), \lim_{|\omega|\to 0} P_\omega(\boldsymbol{u}, \boldsymbol{e}_2)\right),\end{aligned} \tag{5}$$

where $|\omega|$ means the measure of ω.

If the virtual crack extension $\{\Sigma(t)\}_{0\le t\le T}$ is non-smooth at γ and $\lim_{t\downarrow 0}\boldsymbol{J}_{\gamma(t)}(\boldsymbol{u}(t))$ exists , then we have

$$\mathcal{G}(\mathcal{L}, \Omega_{\Sigma(\cdot)}) = \lim_{t\downarrow 0} \boldsymbol{X}_{\gamma(t)}\cdot\boldsymbol{J}_{\gamma(t)}(\boldsymbol{u}(t)). \tag{6}$$

Proof. The mathematical proof of (4) is given in [7] in 2D-case, and the 3D-version of (4) is proved in [8]. Here we notice that $J_\omega(\boldsymbol{u},\boldsymbol{X})$ are independent of ω, and $\lim_{|\omega|\to 0} R_\omega(\boldsymbol{u},\mathcal{X}) = 0$ for arbitrary vector field $\mathcal{X} \in W^{1,\infty}(\mathbb{R}^2)^2$. Then (5) holds. From (5) and by the mean value theorem, we obtain

$$t^{-1}\left[\mathcal{E}(\boldsymbol{u}(0);\mathcal{L},\Omega_{\Sigma(0)}) - \mathcal{E}(\boldsymbol{u}(t);\mathcal{L},\Omega_{\Sigma(t)})\right] = \boldsymbol{X}_{\gamma(\theta t)}\cdot \boldsymbol{J}_{\gamma(\theta t)}(\boldsymbol{u}(\theta t)) \quad \text{with } 0<\theta<1.$$

Since the existence of limit $\lim_{t\downarrow 0}\boldsymbol{J}_{\gamma(t)}(\boldsymbol{u}(t))$ is assumed, we can derive (6). This completes the proof of Theorem 2.3.

Theorem 2.4 *Under the constant loading, if the elasticity is homogeneous isotropic and the crack extension is smooth at γ, then the derivative $-d\mathcal{E}(\boldsymbol{u}(t);\mathcal{L}(t),\Omega_{\Sigma(t)})/dt|_{t=+0}$ depends only on the singularity at the crack tip γ as follows*

$$-\left.\frac{d\mathcal{E}(\boldsymbol{u}(t);\mathcal{L}(t),\Omega_{\Sigma(t)})}{dt}\right|_{t=+0} = \frac{1}{E}\left(K_1(\gamma)^2 + K_2(\gamma)^2\right). \tag{7}$$

If the crack extension is non-smooth at γ and $\lim_{t\downarrow 0} K_i(\gamma(t))$, $i=1,2$ *exists, then*

$$-\left.\frac{d\mathcal{E}(\boldsymbol{u}(t);\mathcal{L}(t),\Omega_{\Sigma(t)})}{dt}\right|_{t=+0} = \lim_{t\downarrow 0}\frac{1}{E}\left(K_1(\gamma(t))^2 + K_2(\gamma(t))^2\right). \tag{8}$$

However, under the varying load $\mathcal{L}(t)$, the derivative $-d\mathcal{E}(\boldsymbol{u}(t);\mathcal{L}(t),\Omega_{\Sigma(t)})/dt|_{t=0}$ does not depend only on the crack extension. By this reason, we introduce another definition of *energy release rate*

$$\mathcal{G}(\mathcal{L}(\cdot),\Omega_{\Sigma(\cdot)}) := \lim_{t\downarrow 0}\frac{1}{2t}\langle \sigma_{ij}(\boldsymbol{u})\nu_j, [\![u_i(t)-u_i]\!]\rangle_{\Sigma(t)}$$

where $\langle\cdot,\cdot\rangle_{\Sigma(t)}$ is the bilinear between the dual space $\left(H^{1/2}_{00}(\Sigma(t))^2\right)'$ and $H^{1/2}_{00}(\Sigma(t))^2$, $\boldsymbol{u}=\boldsymbol{u}(0)$ and $[\![v]\!] = v^+ - v^-$. Since $\sigma_{ij}(\boldsymbol{u})\nu_j \equiv 0$ on Σ and $[\![u_i]\!]\equiv 0$ on $\Sigma(t)\setminus\Sigma$, we think the following formulas will be valid,

$$\langle \sigma_{ij}(\boldsymbol{u})\nu_j, [\![u_i]\!]\rangle_{\Sigma(t)} = 0, \qquad \langle \sigma_{ij}(\boldsymbol{u})\nu_j|_\Sigma, [\![u_i(t)]\!]|_\Sigma\rangle_{\Sigma(t)} = 0.$$

This will derive the integral expression

$$\lim_{t\downarrow 0}\frac{1}{2t}\langle \sigma_{ij}(\boldsymbol{u})\nu_j, [\![u_i(t)-u_i]\!]\rangle_{\Sigma(t)} = \lim_{t\downarrow 0}\frac{1}{2t}\int_{\Sigma(t)\setminus\Sigma}\sigma_{ij}(\boldsymbol{u})\nu_j[\![u_i(t)]\!]d\ell. \tag{9}$$

Theorem 2.5 *If the elasticity is homogeneous isotropic and the crack extension is smooth at γ, then*

$$\lim_{t\downarrow 0}\frac{1}{2t}\int_{\Sigma(t)\setminus\Sigma}\sigma_{ij}(\boldsymbol{u})\nu_j[\![u_i(t)]\!]d\ell = \frac{1}{E}\left(K_1(\gamma)^2 + K_2(\gamma)^2\right). \tag{10}$$

Proof. It will be sufficient to prove in the case when the straight crack extend straightforward near γ. Using the localization technique by the cut-off function and the coordinate transform, we can reduce this problem to the case when the crack lie on the line $\{(x_1,0);\ -\infty < x_1 < 0\}$.

By the Heaviside function $H(x)$, we obtain the expression

$$\begin{aligned}
\begin{bmatrix} \sigma_{22}(x_1, \pm 0) \\ \sigma_{12}(x_1, \pm 0) \end{bmatrix} &= \frac{K_1(\gamma)}{\sqrt{2\pi x_1}} \begin{bmatrix} H(x_1) \\ 0 \end{bmatrix} + \frac{K_2(\gamma)}{\sqrt{2\pi x_1}} \begin{bmatrix} 0 \\ H(x_1) \end{bmatrix} + o(x_1^{-1/2}) H(x_1), \\
\left[\!\left[\begin{matrix} u_1(t; x_1, 0) \\ u_2(t; x_1, 0) \end{matrix} \right]\!\right] &= \frac{(\kappa+1)}{\mu} \sqrt{\frac{t - x_1}{2\pi}} \left(K_1(\gamma(t)) \begin{bmatrix} 0 \\ H(t - x_1) \end{bmatrix} + K_2(\gamma(t)) \begin{bmatrix} H(t - x_1) \\ 0 \end{bmatrix} \right) \\
& \quad + o(\sqrt{t - x_1}) H(t - x_1),
\end{aligned}$$

where $\kappa = (3 - \nu)/(1 + \nu)$ by Poisson's ratio ν. Because

$$\begin{aligned}
\int_{\Sigma(t)\setminus\Sigma} \sigma_{ij}(\boldsymbol{u}) \nu_j [\![u_i(t)]\!] d\ell &= \int_0^t \{ \sigma_{22}(x_1, 0) [\![u_2(t; x_1, 0)]\!] + \sigma_{21}(x_1, 0) [\![u_1(t; x_1, 0)]\!] \} \, dx_1 \\
&= \frac{\kappa + 1}{2\mu\pi} \int_0^t (K_1(\gamma) K_1(\gamma(t)) + K_2(\gamma) K_2(\gamma(t))) \sqrt{(t - x_1)/x_1} dx_1 + o(t) \\
&= \frac{\kappa + 1}{2\mu\pi} \frac{\pi}{2} t \left(K_1(\gamma)^2 + K_2(\gamma)^2 \right) + o(t).
\end{aligned}$$

From this we can derive (10).

But (10) does not hold, if the virtual crack extension is non-smooth at γ.

3 The direction of crack extension

If we apply (10) to the straight initial crack Σ with $\gamma = (0, 0)$ and the virtual kinky crack extension

$$\Sigma_\alpha(t) = \{ (x, y); \; x = l \cos \alpha, \; y = l \sin \alpha, \; 0 \le l \le t \},$$

we then have

$$\int_{\Sigma(t)\setminus\Sigma} \sigma_{ij}(\boldsymbol{u}) \nu_j [\![\boldsymbol{u}_i(t)]\!] d\ell = \int_0^t \{ \boldsymbol{\sigma}_\theta(\boldsymbol{u}(l, \alpha)) [\![u_{\alpha;\theta}(t; t - l, 0)]\!] + \boldsymbol{\sigma}_{l\theta}(\boldsymbol{u}(l, \alpha)) [\![u_{\alpha;r}(t; t - l, 0)]\!] \} \, dl. \quad (11)$$

where

$$\begin{aligned}
\sigma_\theta(\boldsymbol{u}(l, \alpha)) &= \sigma_{11}(\boldsymbol{u}(x)) \sin^2 \alpha + \sigma_{22}(\boldsymbol{u}(x)) \cos^2 \alpha - \sigma_{12}(\boldsymbol{u}(x)) \sin 2\alpha \qquad (12) \\
&= (2\pi l)^{-1/2} (\mathcal{F}_{11}^\alpha K_1(\gamma) + \mathcal{F}_{12}^\alpha K_2(\gamma)) + \sigma_r^R(x), \\
\sigma_{r\theta}(\boldsymbol{u}(l, \alpha)) &= (\sigma_{22}(\boldsymbol{u}(x)) - \sigma_{11}(\boldsymbol{u}(x))) \sin \theta \cos \theta + \sigma_{12}(\boldsymbol{u}(x)) \cos 2\theta \\
&= (2\pi l)^{-1/2} (\mathcal{F}_{21}^\alpha K_1(\gamma) + \mathcal{F}_{22}^\alpha K_2(\gamma)) + \sigma_{r\theta}^R(x), \\
\mathcal{F}_{11}^\alpha &= \tfrac{3}{4} \cos(\alpha/2) + \tfrac{1}{4} \cos(3\alpha/2), \; \mathcal{F}_{12}^\alpha = -\tfrac{3}{4} \sin(\alpha/2) - \tfrac{3}{4} \sin(3\alpha/2), \\
\mathcal{F}_{21}^\alpha &= \tfrac{1}{4} \sin(\alpha/2) + \tfrac{1}{4} \sin(3\alpha/2), \; \mathcal{F}_{22}^\alpha = \tfrac{1}{4} \cos(\alpha/2) + \tfrac{3}{4} \cos(3\alpha/2),
\end{aligned}$$

$u_{\alpha;r}(t; r_t, \theta_t)$, $u_{\alpha;\theta}(t; r_t, \theta_t)$ the compornent of $\boldsymbol{u}(t)$ in the polar coordinate (r_t, θ_t) of the center $\gamma_\alpha(t)$ and $\sigma_\theta^R, \sigma_{r\theta}^R \in H^1(\text{near } \gamma)$. The jumps $[\![u_{\alpha;\theta}(t; r_t, 0)]\!]$, $[\![u_{\alpha;r}(t; r_t, 0)]\!]$ express the opening and sliding displacements of the crack surface, respectively. Then, near the crack tip $\gamma_\alpha(t) = (t\cos\alpha, t\sin\alpha)$, we have the expression

$$\begin{aligned} [\![u_{\alpha;\theta}(t;t-l,0)]\!] &= K_1(\gamma_\alpha(t))(2\mu)^{-1}\sqrt{(t-l)/(2\pi)} + o(\sqrt{t-l}), \\ [\![u_{\alpha;r}(t;t-l,0)]\!] &= K_2(\gamma_\alpha(t))(2\mu)^{-1}\sqrt{(t-l)/(2\pi)} + o(\sqrt{t-l}) \end{aligned} \tag{13}$$

on $\Sigma_\alpha(t)$. If $K_{i,\alpha}(\gamma) = \lim_{t\downarrow 0} K_i(\gamma_\alpha(t))$ exist for $i = 1, 2$, then by the mean value theorem, we obtain

$$K_1(\gamma_\alpha(t)) = K_{i,\alpha}(\gamma) + K_i'(\gamma_\alpha(\tau))t, \quad \text{for } 0 < \tau < t. \tag{14}$$

Combining (11)–(14), we can derive that the left-hand side of (11) is

$$\frac{1}{E}\left\{(\mathcal{F}_{11}^\alpha K_1(\gamma) + \mathcal{F}_{21}^\alpha K_2(\gamma))K_{1,\alpha}(\gamma) + (\mathcal{F}_{21}^\alpha K_1(\gamma) + \mathcal{F}_{22}^\alpha K_2(\gamma))K_{2,\alpha}(\gamma)\right\}. \tag{15}$$

On the other hand, the left-hand side of (11) becomes by (8)

$$\frac{1}{E}\left(K_{1,\alpha}(\gamma)^2 + K_{2,\alpha}(\gamma)^2\right). \tag{16}$$

Therefore, it is possible that

$$K_{i,\alpha}(\gamma) = \mathcal{F}_{i1}^\alpha K_1(\gamma) + \mathcal{F}_{i2}^\alpha K_2(\gamma) \tag{17}$$

for $i = 1, 2$. However, **(17) is not valid** by the papers [14, 12, 1] they have been shown in engineering general stuations

$$\begin{aligned} K_{i,\alpha}(\gamma) &= F_{i1}^\alpha K_1(\gamma) + F_{i2}^\alpha K_2(\gamma) \quad \text{for } i = 1, 2, \\ F_{11}^\alpha &= 1 - \frac{3\pi^2}{8}m^2 + \left(\pi^2 - \frac{5\pi^4}{128}\right)m^4 + \left(\frac{\pi^2}{9} - \frac{11\pi^4}{72} + \frac{119\pi^6}{15360}\right)m^6 + O(m^8), \\ F_{12}^\alpha &= -\frac{3\pi}{2}m + \left(\frac{10\pi}{3} + \frac{\pi^3}{16}\right)m^3 + \left(-2\pi - \frac{133\pi^3}{180} + \frac{59\pi^5}{1280}\right)m^5 + O(m^7), \\ F_{21}^\alpha &= \frac{\pi}{2}m - \left(\frac{4\pi}{3} + \frac{\pi^3}{48}\right)m^3 + \left(-\frac{2\pi}{3} + \frac{13\pi^3}{30} - \frac{59\pi^5}{3840}\right)m^5 + O(m^7), \\ F_{22}^\alpha &= 1 - \left(4 + \frac{3\pi^2}{8}\right)m^2 + \left(\frac{8}{3} + \frac{29\pi^2}{18} - \frac{5\pi^4}{128}\right)m^4 \\ &\quad + \left(-\frac{32}{15} - \frac{4\pi^2}{9} - \frac{1159\pi^4}{7200} + \frac{119\pi^6}{15360}\right)m^6 + O(m^8), \end{aligned} \tag{18}$$

with $m = \alpha/\pi$. The taylor's expansions of $\mathcal{F}_{ij}^\alpha$ $(i, j = 1, 2)$ are

$$\begin{aligned} \mathcal{F}_{11}^\alpha &= 1 - \frac{3\,m^2\,\pi^2}{8} + \frac{7\,m^4\,\pi^4}{128} - \frac{61\,m^6\,\pi^6}{15360} + O(m^7) \\ \mathcal{F}_{21}^\alpha &= \frac{-3\,m\,\pi}{2} + \frac{7\,m^3\,\pi^3}{16} - \frac{61\,m^5\,\pi^5}{1280} + O(m^7), \\ \mathcal{F}_{12}^\alpha &= \frac{m\,\pi}{2} - \frac{7\,m^3\,\pi^3}{48} + \frac{61\,m^5\,\pi^5}{3840} + O(m^7), \\ \mathcal{F}_{22}^\alpha &= 1 - \frac{7\,m^2\,\pi^2}{8} + \frac{61\,m^4\,\pi^4}{384} - \frac{547\,m^6\,\pi^6}{46080} + O(m^7). \end{aligned}$$

This leads the following modification of (17)

$$K_{i,\alpha}(\gamma) = \mathcal{F}^{\alpha}_{i1} K_1(\gamma) + \mathcal{F}^{\alpha}_{i2} K_2(\gamma) + O(\alpha^2). \tag{19}$$

There are many criterions which determine the crack direction α^*. We only show the famous three criterions in homogeneous isotropic elastic plate.

Definition 3.1 (maximun stress criterion [2]) *Find the angle α^* that is the maximum value of $\mathcal{F}^{\alpha}_{11} K_1(\gamma) + \mathcal{F}^{\alpha}_{12} K_2(\gamma)$ over $-\pi < \alpha < \pi$.*
Also α^* satisfy $\mathcal{F}^{\alpha^*}_{21} K_1(\gamma) + \mathcal{F}^{\alpha^*}_{22} K_2(\gamma) = 0$.

Definition 3.2 (maximun energy release rate (see for example [14, 1])) *Find the angle α^* that is the maximum value of the energy release rate equivalent to $\frac{1}{E}\left(K_{1,\alpha}(\gamma)^2 + K_{2,\alpha}(\gamma)^2\right)$.*

Definition 3.3 (local symmetry (see for example [14, 12, 1])) *Find the angle α^* that satisfy the condition $K_{2,\alpha^*}(\gamma) = 0$.*

If (17) is true, we can prove the following; If the crack extends smoothly, then $\alpha^* = 0$ and all criterions are valid. But each criterions make the difference when α^* far from 0 (see Figure 2).

We now apply the various criterions of the direction for an infinite body loaded by uniform forces at infinity with the angle β. By the experiment, the crack extends straightforward at $\beta = \pi/2 \sim 1.57$. The Figure 2 indicates that all criterions give the similar angles near $\beta = \pi/2$, but they differ near $\beta = 0$. The curve with the label $J^2_\alpha(\gamma)$ is the same one by the maximum energy release rate criterion, and the curve with $J^1_\alpha(\gamma)$ is obtained by the formal application of J-integral for non-smooth crack extension. In the calculation, we used (18).

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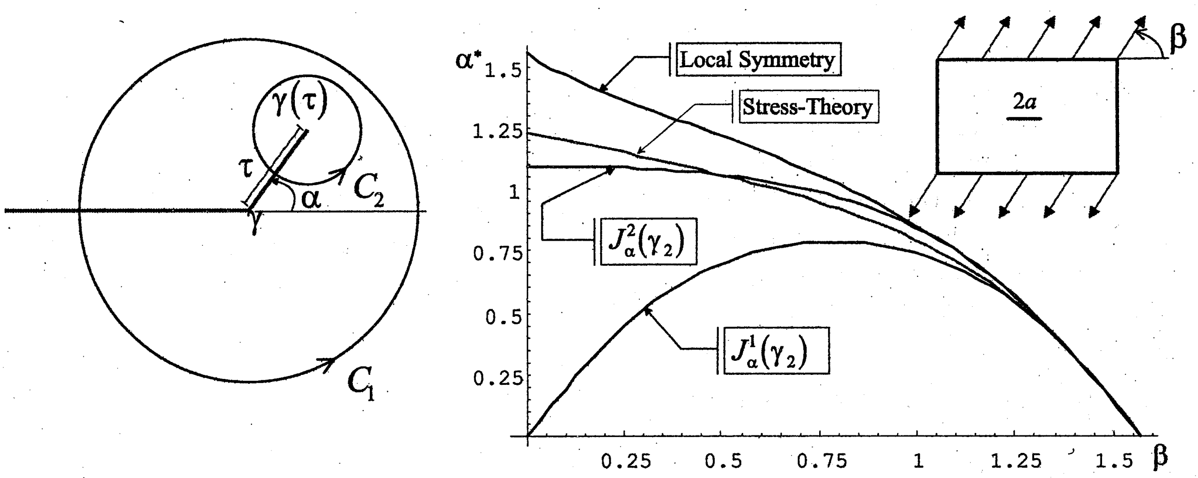

$$
\begin{aligned}
J^1_\alpha(\gamma) &= J_{C_1}(\boldsymbol{u}, \boldsymbol{e}_1)\cos\alpha + J_{C_1}(\boldsymbol{u}, \boldsymbol{e}_2)\sin\alpha, \\
J^2_\alpha(\gamma) &= \lim_{\tau\to 0}\lim_{C_2\to 0}\left[J_{C_2}(\boldsymbol{u}(\tau), \boldsymbol{e}_1)\cos\alpha + J_{C_2}(\boldsymbol{u}(\tau), \boldsymbol{e}_2)\sin\alpha\right].
\end{aligned}
$$

Figure 2: The direction of crack extension by various criterion.